

# Nonlinear electron dynamics in a rippled channel with time-dependent electric field: Quantum Arnol'd diffusion

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## Abstract

We study the electron dynamics in a 2D waveguide bounded by a periodically rippled surface in the presence of the time-periodic electric field. The main attention is paid to a possibility of a weak quantum diffusion along the coupling resonance, that can be associated with the classical Arnol'd diffusion. It was found that quantum diffusion is possible only when the perturbation is large enough in order to mix many near-separatrix levels. The rate of the quantum diffusion turns out to be less than the corresponding classical one, thus indicating the influence of quantum coherent effects. Another important effect is the dynamical localization of the quantum diffusion, that may be compared with the famous Anderson localization occurring in 1D random potentials. Our estimates show that the quantum Arnol'd diffusion can be observed in semi-metal rippled channels, for which the scattering and decoherence times are larger than the saturation time due to the dynamical localization.

# 1 Introduction

In this paper we consider the quantum dynamics of a particle in two-dimensional rippled channel subjected to time-dependent electric field. In the absence of the electric field, the particle motion can be either regular or weakly/strongly chaotic, depending on the model parameters. In the case of regular motion, the linear response to an external electromagnetic field can be described by the famous Kubo formula. In the opposite case of a strong chaos one can use the approach recently developed in Refs. [1, 2]. The situation with weak chaos is quite peculiar and needs a special approach. Of specific interest is the case when chaos is extremely weak, and occurs in narrow regions of slightly destroyed non-linear resonances, thus leading to the so-called Arnol'd diffusion *along* the resonances [3, 4]. Our further analysis enlightens the connection between quantum and classical mechanisms of the Arnol'd diffusion on the rippled channel model. The results presented here are based on the theory developed in Ref. [5], and expand those preliminary reported in Ref. [6].

The dynamical chaos in classical Hamiltonian systems is related to the destruction of separatrices of nonlinear resonances [7]. For a weak interaction, chaotic motion occurs only in the vicinity of separatrices of the resonances. On the other hand, inside the resonances the motion remains regular, in accordance with the Kolmogorov-Arnol'd-Moser (KAM) theory (see, for example, Ref. [4]). If the number  $N$  of degrees of freedom larger than 2, the KAM surfaces do not separate the stochastic layers, therefore, they form a stochastic web that can cover whole phase space of a system. Thus, if trajectory starts *inside* the stochastic web, it can diffuse throughout the phase space. Such a diffusion along stochastic webs was predicted by Arnol'd in 1964 [8], and since that time it is known as an universal mechanism for instability and chaos in generic nonlinear Hamiltonian systems with  $N > 2$  (see, for example, review [3] and references therein).

The chaotic dynamics of a quantum systems under time dependent periodic perturbation was studied in a series of works. The authors of Ref. [1] in the frame of random matrix theory have investigated the spectral properties of evolution operator in the generic system with a time-dependent Hamiltonian. The statistical properties of quasienergy spectrum, the localization of the eigenstates of evolution operators as well as the process of saturation of the energy absorption in the external periodic field was considered. In the paper [2] the response of the quantum dot electron system to a periodic perturbation was studied analytically in terms of zero-dimensional

time-dependent nonlinear sigma model. The quantum correction to the energy absorption rate as a function of the dephasing time was calculated. In particular, it was shown that the dynamical localization corrections similar to the  $d$ -dimensional weak localization corrections to conductivity if the perturbation is a sum of  $d$  incommensurate harmonic functions. A typical application of these results would be to the response of a different electron mesoscopic systems (quantum wells, wires and dots) to electromagnetic radiation. Experimentally the response of 2D electron gas in quantum dot formed in GaAs/AlGaAs heterojunction to electromagnetic radiation was investigated in [9, 10], where the effect of absorption saturation in open chaotic quantum dots was observed in particular.

The chaotic nature of the free particle dynamics in a rippled channel (without external electromagnetic fields) has been investigated in Refs. [11, 12, 13, 14], both in classical and quantum models. In particular, in Ref. [11] the transport properties were considered in a ballistic regime. The energy band structure, eigenfunctions and density of states have been analyzed in Refs. [12, 13]. The structure of quantum states in the channel with rough boundaries, including the phenomena of quantum localization, have been studied in Ref. [14]. In particular, it was found that the eigenstates are very different in their localization properties.

The paper is organized as follows. In Sec. 2 we discuss the classical model and the mechanism of the Arnol'd diffusion in the rippled channel. Stationary quantum states corresponding to the coupling resonance are studied in Sec. 3. In Sec. 4 we build the evolution operator for one period of the external field, and discuss its structure. In the same section both classical and quantum diffusion coefficients are calculated as a function of the goffer amplitude. The nature of the quantum diffusion suppression, as well as the dynamical localization, are also discussed here. In last section we formulate main results obtained in the paper, and discuss the parameters for which the quantum Arnol'd diffusion could be observed experimentally.

## 2 Classical Arnol'd diffusion in 2D rippled channel

We study the Arnol'd diffusion in a periodic two-dimensional waveguide, see Fig. 1. It is defined by the upper profile given in dimensionless variables by

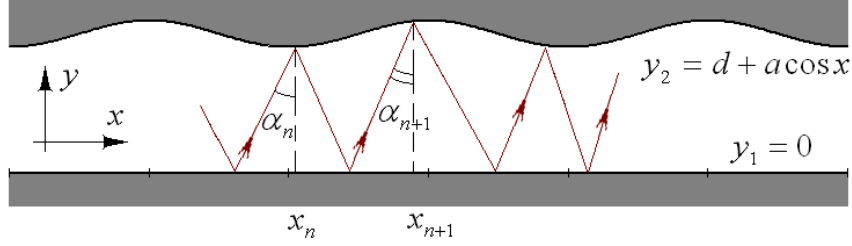


Figure 1: An example of the electron trajectory in the rippled channel.

the function  $y = d + a \cos x$  and by the low profile that is assumed to be flat,  $y = 0$ . Here  $d$  is the average width,  $a$  is the ripple amplitude, and the goffer period is equal to  $2\pi$ . In what follows we put the ratio  $a/d$  is small, in order to avoid the global chaos that occurs for  $a/d \sim 1$ . The collisions of a particle with these boundaries are assumed to be elastic.

The Poincaré map for the positions and angles  $(x_n, \alpha_n)$  of the reflections from the upper wall is determined by

$$\begin{cases} \alpha_{n+1} = \alpha_n - 2 \arctan(a \sin x_n), \\ x_{n+1} = x_n + \tan \alpha_{n+1} (2d + a(\cos x_n + \cos x_{n+1})), \end{cases} \quad (1)$$

Here  $x_n$  is the position corresponding to the  $n$ -th bounce at the rippled wall, and  $\alpha_n$  is the angle of the particle trajectory makes with the vertical at  $x = x_n$ .

An example of the trajectory described by this map is shown in Fig. 2(a). It can be shown that there are many nonlinear resonances arising in this model due to the coupling between two degrees of freedom, with the resonance condition,  $\eta = T_x/T_y = \omega_y/\omega_x$ . Here  $T_y$  is the period of a transverse oscillation inside the channel,  $T_x$  is the time of flight of a particle over one period of the waveguide,  $\omega_x$  and  $\omega_y$  are the corresponding frequencies, and  $\eta$  is the rational number. It should be noted that in the neighborhood of the resonances  $\eta = 1/n$  (for which  $n = 0, 1, 2, \dots$ ) it is possible to reduce the map (1) to the well known standard map [3] with parameter  $K_n = 4ad(1 + (\pi n/d)^2)$ .

The mechanism of the classical Arnol'd diffusion in this system is illustrated in Fig. 2(b). Here some of the resonance lines for the different values of  $\eta$  shown in Fig. 2(a), are presented with the use of the  $\omega_x, \omega_y$ -plane. The curve of a constant kinetic energy  $E$  is also shown here, determined by the

equation

$$\omega_x^2 + \left( \frac{\omega_y d}{\pi} \right)^2 = \frac{2E}{m}, \quad (2)$$

where  $m$  is the particle mass that we will set to unity.

As one can see, there are two different types of resonances. The resonances with  $\eta \leq 1/3$  are overlapped and in this region the global chaos regime is realized. All other resonances are not overlapped, and surrounded by narrow near-separatrix stochastic regions. These resonances are isolated one from another by the KAM-surfaces, and therefore, for a weak perturbation the transition between their stochastic layers is forbidden. Such a transition could occur in the case of the resonance overlap only. In the absence of an external field, the passage of a trajectory *along* any stochastic layer (this direction is shown in Fig. 2(b) for resonance  $\eta = 1$  by two arrows) is also impossible because of the energy conservation. The external time-periodic field removes the latter restriction, and slow diffusion along stochastic layers becomes possible. As a result, the particle remains located on the coupling resonance, however, with a proportional change of its momentum components.

The external electric field is given by the potential  $V(y, t) = -f_0 y (\cos \Omega_1 t + \cos \Omega_2 t)$ , giving rise to two main resonances,  $\omega_y = \Omega_1$  and  $\omega_y = \Omega_2$ . In order to calculate the diffusion rate, we consider a part of the Arnol'd stochastic web created by three resonances, namely, by the coupling resonance  $\omega_x = \omega_y$  and two driving resonances with frequencies  $\Omega_1$  and  $\Omega_2$ . Correspondingly, we choose the initial conditions inside the stochastic layer of the coupling resonance. To avoid a strong overlap of the resonances, however, to provide a weak chaos in the near-separatrix region, we assume that the relation  $a/f_0 = 10^{-3} \ll 1$  is fulfilled.

### 3 Stationary states at coupling resonance

In the following it is convenient to rewrite the problem in the curvilinear coordinates  $\tilde{x}^i$  for which the both channel boundaries appear to be flat [15]. As a result, the covariant coordinate representation of the Schrödinger equation has the form

$$-\frac{1}{2\sqrt{g}} \frac{\partial}{\partial \tilde{x}^i} \sqrt{g} g^{ij} \frac{\partial \psi}{\partial \tilde{x}^j} = E\psi \quad (3)$$

where  $g^{ij}$  is the metric tensor and  $g \equiv \det(g_{ij})$ . Here we use the units in which the Plank's constant and effective mass are equal to unity. Let the new coordinates are given by the relations,

$$\tilde{x} = x, \quad \tilde{y} = \frac{y}{1 + \epsilon \cos x} \quad (4)$$

where  $\epsilon = a/d$ . Then the boundary conditions read as  $\psi(\tilde{x}, 0) = \psi(\tilde{x}, d) = 0$  and the metric tensor is

$$g^{ij} = \begin{pmatrix} 1 & \frac{\epsilon \tilde{x} \sin \tilde{x}}{1 + \epsilon \cos \tilde{x}} \\ \frac{\epsilon \tilde{x} \sin \tilde{x}}{1 + \epsilon \cos \tilde{x}} & \frac{1 + \epsilon^2 \tilde{x}^2 \sin^2 \tilde{x}}{(1 + \epsilon \cos \tilde{x})^2} \end{pmatrix}, \quad (5)$$

with the orthonormality condition,

$$\int \psi_i^* \psi_j \sqrt{g} d\tilde{x} d\tilde{y} = \delta_{ij}. \quad (6)$$

If the ripple amplitude  $a$  is small compared to the channel width  $d$ , one can safely keep only the first-order terms in  $\epsilon$  in the Schrödinger equation (3). This strongly simplifies numerical simulations without the loss of generality. However, one should note that with an increase of roughness in the scattering profile  $y(x)$  this approximation may be invalid, due to an influence of the so-called gradient scattering, see details in Refs.[14]. As a result, we obtain the following Hamiltonian [15],

$$\begin{aligned} \hat{H} = \hat{H}_0 + \epsilon \hat{U} = & -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \\ & + \frac{\epsilon}{2} \left( 2 \cos x \frac{\partial^2}{\partial y^2} - 2y \sin x \frac{\partial^2}{\partial x \partial y} - y \cos x \frac{\partial}{\partial y} - \frac{1}{2} \cos x - \sin x \frac{\partial}{\partial x} \right). \end{aligned} \quad (7)$$

Here and below we omitted tildes in coordinates  $\tilde{x}$  and  $\tilde{y}$ .

Since the Hamiltonian is periodic in the longitudinal coordinate  $x$ , the eigenstates are Bloch states characterized by the Bloch index  $k$ . Therefore, the eigenstates  $\psi^k(x, y)$  of the total Hamiltonian  $\hat{H}$  can be presented as follows,

$$\psi^k(x, y) = \sum_{n,m} c_{nm}^k \psi_{nm}^0(k, x, y) \quad (8)$$

where

$$\psi_{nm}^0(k, x, y) = \frac{1}{\sqrt{\pi d}} e^{i(n+k)x} \sin\left(\frac{\pi m y}{d}\right) \quad (9)$$

are the eigenstates of the unperturbed Hamiltonian  $\hat{H}_0$ . Here we normalize wave functions to the length  $L = 2\pi$  of the period in  $x$ -direction. In the absence of the perturbation ( $\epsilon = 0$ ) the energy spectrum has the form

$$E_{nm}^0(k) = \frac{1}{2} \left( (n+k)^2 + \frac{\pi^2 m^2}{d^2} \right). \quad (10)$$

The Bloch wave vector  $k$  has continuous values, in particular,  $-1/2 \leq k \leq 1/2$  in the first Brillouin zone.

Now we proceed to solve the system of algebraic equations for the coefficients  $c_{nm}^k$ ,

$$E(k)c_{nm}^k = E_{nm}^0(k)c_{nm}^k + \epsilon \sum_{n',m'} U_{nm,n'm'}^k c_{n'm'}^k, \quad (11)$$

with  $-\infty < n < \infty$  and  $m = 1, \dots, \infty$ . The matrix elements in (11) are defined as

$$U_{nm,n'm'}^k = \int (\psi_{(k+n'),m'}^0)^* \hat{U}(x,y) \psi_{(k+n),m}^0 dx dy = -\frac{1}{2} \left[ \frac{\pi^2 m^2}{d^2} (\delta_{n',n+1} + \delta_{n',n-1}) \delta_{m,m'} \right. \\ \left. + \frac{(-1)^{m+m'}}{m^2 - m'^2} \left( (1 + 2(k+n)) \delta_{n',n+1} + (1 - 2(k+n)) \delta_{n',n-1} \right) \right]. \quad (12)$$

Following Refs. [5], we analyze the energy spectrum in the vicinity of the main coupling resonance  $\eta = 1$  determined by the condition  $\omega_{n_0} = \omega_{m_0}$  with  $\omega_{n_0} = E_{n_0+1}(k) - E_{n_0}(k) = k + n_0 + 1/2$  and  $\omega_{m_0} = E_{m_0+1} - E_{m_0} = \pi^2(2m_0 + 1)/2d^2$ . In a deep semiclassical region for  $n_0 \gg 1$  and  $m_0 \gg 1$ , one can write  $\omega_{n_0} \approx n_0$  and  $\omega_{m_0} \approx \pi^2 m_0/d^2$ . It should be noted that the similar resonance condition can be satisfied for negative  $n_0$  as well, when  $-n_0 \approx \pi^2 m_0/d^2$ , that corresponds to the motion in the opposite direction. Since below we assume large values,  $|n| \gg 1$ , one can neglect the tunneling from the resonances with  $n > 0$  to those with  $n < 0$ . Note also that for the values of  $k$  at the center of the energy band,  $k = 0$ , as well as at the band edges,  $k = \pm 1/2$ , there are additional integrals of motion due to anti-unitary symmetry. For this reason in what follows we consider generic case of other values of  $k$ . The properties of the energy spectra and eigenstates for specific values  $k = \pm 1/2$  and  $k = 0$  will be discussed separately.

In the vicinity of the coupling resonance it is convenient to introduce new indexes  $r = n - n_0$  and  $p = r + (m - m_0)$ . In this notation instead of the

system (11) we obtain,

$$E(k)c_{rp}^k = \left( p\omega_{m_0} + \frac{r^2}{2} + \frac{\pi^2}{2d^2}(p-r)^2 + kr \right) c_{rp}^k + \sum_{r',p'} U_{rp,r'p'}^k c_{r'p'}^k, \quad (13)$$

where energy  $E(k)$  counts from the level  $E_{n_0 m_0}^0(k)$ .

With equation (13) one can investigate quantum states at the coupling resonance for different values of the wave number  $k$ . For example, by considering the states near some positive value  $n_0$  and for  $-1/2 < k < 1/2$  (with  $k \neq 0$ ), we obtain that the energy spectrum consists of series of Mathieu-like groups. These groups are separated one from another by the energy  $\omega_{n_0}$ . The structure of energy spectrum in each group is typical for any quantum non-linear resonance. Namely, the lowest levels are practically equidistant, and the accumulation point corresponds to the classical separatrix. However, in contrast with the well known Mathieu spectrum, all the above-separatrix states are non-degenerate even for the zero goffer  $a = 0$ .

Besides the above discussed part of the spectrum, another series of groups occurs in the same energy region, that corresponds to the similar resonance with the negative values of  $n_0$ . The eigenstates corresponding to this resonance are the waves propagating in the negative direction of the  $x$  axis.

In accordance with the spectrum structure it is convenient to characterize the states at the coupling resonance by two indexes: the group number  $q$ , and level number  $s$  characterizing the levels inside the group. Correspondingly, the energy of each group can be written in the form,

$$E_{q,s}(k) = \omega_{n_0}(k)q + E_{q,s}^M(k), \quad (14)$$

where  $E_{q,s}^M(k)$  is the Mathieu-like spectrum for one group. The indexes  $q$  and  $s$  correspond to fast and slow variables characterizing the motion inside the classical coupling resonance. Fig. 3 illustrates the structure of the spectrum for one group with  $q = 0$ , in dependence on  $k$ . The states corresponding to  $n_0 > 0$  shown in the figure, have positive values of the derivative  $\partial E / \partial k$ .

Let us discuss now the resonance states for specific values  $k = 0$  and  $k = \pm 1/2$ . The corresponding states have no classical counterparts because the waves propagating in opposite directions are coupled due to the Bragg condition. Indeed, at the center and at the edges of the Brillouin zone, where the group velocity vanishes,  $\partial E / \partial k = 0$ , the Fourier series (8) of the Bloch function  $\psi^k(x, y)$  contains the terms with positive and negative  $n$  equally.



As a result, the Bloch function  $\psi^k(x, y)$  is the even (and real) or odd (and pure imaginary) standing wave. In contrast, the Bloch functions for other values of  $k$  have no definite parity and can be characterized by a positive or negative momentum.

As one can see, in Eqs. (8) and (9) for  $k = 0$  it is more convenient to use the functions  $\cos nx$  and  $\sin nx$ , instead of the exponents  $e^{inx}$ . Then, the matrix elements  $U_{nm, n'm'}^k$  are not different from the analogous ones (12), and, therefore, the spectrum structure is similar to that discussed before for generic values of  $k$ . In a similar way, the functions  $\cos(n \pm 1/2)x$  and  $\sin(n \pm 1/2)x$  should be used in the expansion (8) for  $k = \pm 1/2$ .

The resonance states for  $k = 0$  and  $k = \pm 1/2$  are either even or odd, therefore, the energy spectrum consists of two series of Mathieu-like groups corresponding to a specific parity. For  $k = 0$  the above-separatrix levels in each group are practically non-degenerated, in contrast with the case of generic values of  $k$ , see Fig. 3.

The wave functions of the resonance quantum states have very rich and complex structure. To make it clear we plotted the probability distribution of the eigenvectors of the Eq. (13) with different  $q$  and  $s$  in the unperturbed basis (9) of the Hamiltonian  $\hat{H}_0$ , see Fig. 4. The probability distribution  $|\psi_{q,s}^k(x, y)|^2$  in  $(x, y)$ -space for the same states are shown at Fig. 5. First, let us discuss the results shown in Fig. 4. Here and below, the ground states in each group have the number  $s = 0$ , and all other states are reordered according to the energy increase, labelling by  $1, -1, 2, -2, \dots$  etc.

So, in Fig. 4(a) one can see probability distribution  $|c_{rp}^k|^2$ , corresponding to the lowest level in the group with  $q = 0$ . In Figs. 4(b,c) the probability distributions corresponding to the 10th and 20th resonance levels of the same group are depicted. To compare with, we show in Fig. 4(d) the distribution corresponding to the level  $s = -18$  taken from the near-separatrix region. All these states are symmetric with respect to  $n - n_0 = 0$  and  $m - m_0 = 0$ . It is seen that the degree of delocalization of these eigenfunctions in the diagonal direction increases with an increase of the energy. Specifically, the separatrix state is much more extended in the unperturbed basis, in comparison with the state at the resonance center. As will be shown below, the separatrix resonance states, having a maximal variance in the unperturbed basis, provide the Arnol'd diffusion. The eigenfunctions corresponding to two near-degenerate levels above the separatrix (with  $s = \pm 41$ ), are shown in Fig. 4(e,f). In contrast, the probability distributions corresponding to above-separatrix states are non-symmetric (their maximums are shifted with

respect to the origin), and have a relatively small variance.

Fig. 5 illustrates the probability distribution for same resonance eigenstates in the transformed  $(x, y)$ -coordinates. Note that for small values  $a/d \ll 1$  the probability distribution in original coordinates has a practically similar shape. It is clear that the structure of these states is quite different from the unperturbed ones. A regular pattern in Fig. 5(a), plotted for the state  $s = 0$ , resembles the classical trajectory corresponding to the resonance center. This orbit starts at point  $x = -\pi$ ,  $y = 0$  and ends at  $x = \pi$ ,  $y = 0$ . The reflection point is located at  $x = 0$ ,  $y = \pi$  where the width of the channel has a maximum. In accordance with the uncertainty principle, this distribution has a nonzero width. The probability distributions in Figs. 5(b,c), corresponding to the 10th and 20th resonance states, also resemble near-resonance classical trajectories, however, have a complex internal structure. Each state of this kind corresponds to the group of classical trajectories in  $(x, y)$ -space, for which initial conditions lay at one closed orbit in the phase space  $(\sin \alpha, x)$ . One can see that the number of white longitudinal lines (minima) are equal to the number of resonance state minus one. Fig. 5(d) illustrates the probability distribution for one of the near-separatrix states,  $s = -18$ . Here the pattern also originated from classical unstable trajectory. It should be stressed that for all presented states the probability distributions mainly have a non-chaotic character. In Fig. 5(e,f) we demonstrate the probability distributions for above-separatrix states  $s = \pm 41$ . The distributions for these states are uniform, indicating that they are weakly perturbed. In conclusion, we would like to point out that the electron density distribution shown in Figs. 5 resemble the electromagnetic field distribution in quasi-optic regimes [16].

Using the same technique we have calculated also quantum states for the resonances  $\eta = 1/2$  and  $\eta = 1/3$ . These resonances are somewhat narrow than  $\eta = 1$  (see Fig. 2(b)) and the number of under-separatrix levels are smaller. As an example Fig. 6 illustrates the probability distributions  $|\psi(x, y)|^2$  for the resonance centers and near-separatrix states for  $\eta = 1/2$  ((a) and (b)) and  $\eta = 1/3$  ((c) and (d)) correspondingly. It is clear that their structures are also reflect the character of correspondent classical trajectories at these resonances.

## 4 Evolution operator

In this section we consider the dynamics of a charged particle in the rippled channel in the presence of the time-dependent electric field described by the potential  $V(y, t) = -f_0 y (\cos \Omega_1 t + \cos \Omega_2 t)$ . We assume that the frequencies  $\Omega_1$  and  $\Omega_2$  are chosen to fulfill the condition  $\omega_{n_0} = (\Omega_1 + \Omega_2)/2$  in order to provide equal driving forces for a particle inside the stochastic layer of the separatrix at the main coupling resonance. Specifically, we take,  $\omega_{n_0} = 400$ ,  $\Omega_1 = 350$ ,  $\Omega_2 = 450$ , therefore, the period  $T$  of the perturbation is,  $T = 7 \cdot 2\pi/\Omega_1 = 9 \cdot 2\pi/\Omega_2 \approx 0.126$ .

Since the total Hamiltonian is periodic in time, one can write the solution of the non-stationary Schrödinger equation as  $\psi_Q(x, y, t) = \exp(-i\epsilon_Q t) u_Q(x, y, t)$ , where  $u_Q(x, y, t + T) = u_Q(x, y, t)$  is the quasienergy (QE) function and  $\epsilon_Q$  is the quasienergy. It can be shown that the QE functions are the eigenfunctions of the evolution operator  $\hat{U}(T)$  of the system for one period of the perturbation. The procedure to determine this operator was described in details in Ref. [5]. The matrix elements  $U_{q,s,q',s'}(T)$  of the evolution operator can be calculated by means of the numerical solution of the non-stationary Schrödinger equation for different initial states. The eigenvectors  $A_{q,s}^Q$  (in the representation of stationary problem with eigenfunctions (8)) and eigenvalues  $\exp(-i\epsilon_Q T)$  can be found numerically by a direct diagonalization of the evolution operator matrix. After that, the evolution matrix  $U_{q,s,q',s'}(NT)$  for  $N$  periods can be written as

$$U_{q,s;q',s'}(NT) = \sum_Q A_{q,s}^Q A_{q',s'}^{Q*} \exp(-i\epsilon_Q NT). \quad (15)$$

Therefore, the evolution of any initial state can be computed by making use of the evolution matrix,

$$C_{q,s}(NT) = \sum_{q',s'} U_{q,s,q',s'}(NT) C_{q',s'}(0). \quad (16)$$

In order to illustrate the structure of matrix  $U_{q,s,q',s'}(T)$  we plot the matrix elements modules in Fig. 7. As one can see, this matrix has global block structure, with a cross-like structure at the centers of blocks. When the evolution operator acts on the initial state  $C_{q,s} = \delta_{q,q_0} \delta_{s,s_0}$ , the resultant state will coincide with the  $(q_0, s_0)$ -column of the matrix. The edges of “crosses” at the block’s centers where matrix elements are relatively large, corresponds

to a transition between separatrix states. As a result, the transition between such states of neighbor groups (along the coupling resonance) is much stronger than those between other states. Note that the separatrix states are responsible for the quantum Arnol'd diffusion.

Our goal is to analyze the dynamics of a particle, initially placed inside the separatrix under the condition that the coupling and two driving resonances do not overlap. In Fig. 8 typical dependencies of the variance  $\Delta_q = \overline{(\Delta\bar{H})^2}/\omega_{n_0}^2$  of the normalized energy are shown versus the time measured in the number  $N$  of periods of the external perturbation, for different initial conditions. Here the quantity  $\Delta_q$  is defined as the variance of a wave packet in the  $q$ -space,  $\Delta_q = \sum_q (q - \bar{q})^2 \sum_s |C_{q,s}|^2$ , where  $\bar{q} = \sum_q q \sum_s |C_{q,s}|^2$ .

The data presented at Fig. 8 clearly demonstrate a different character of the evolution of the system in dependence on initial state. For the state taken from the center of the coupling resonance, as well as from above the separatrix, the variance oscillates in time, in contrast with the state taken from inside the separatrix. In the latter case, after a short time the variance of the energy increases linearly in time, thus manifesting a diffusion-like spread of the wave packet.

In order to characterize the speed of the diffusion, we have calculated classical diffusion coefficients  $D_1$ ,  $D_{1/2}$  and  $D_{1/3}$  for the resonances  $\eta = 1$ ,  $\eta = 1/2$  and  $\eta = 1/3$ , correspondingly:

$$D = \overline{\left( \frac{(\bar{E}_{i+1} - \bar{E}_i)^2}{(\Delta t_i + \Delta t_{i+1})/2} \right)}. \quad (17)$$

To suppress large fluctuations of the energy in time and to reveal a stochastic character of motion, the averaging in Eq. (17) was performed in two stages. Here  $\bar{E}_i$  is the average value of the particle energy over the time  $\Delta t_i$  corresponding to  $10^3$  collisions with the rippled wall. The second average has been done in the following way. Having the mean value  $\bar{E}_i$  in each interval  $\Delta t_i$ , the difference  $(\bar{E}_{i+1} - \bar{E}_i)$  between adjacent intervals was computed, and after that the expression in brackets in Eq. (17) was averaged over all these differences.

The quantum diffusion coefficient  $D_{1q}$  was calculated only for the resonance  $\eta = 1$ , see Fig. 9. It was found that the quantum Arnol'd diffusion roughly corresponds to the classical one. However, the data clearly indicate that the quantum diffusion is systematically weaker than the classical Arnol'd diffusion. In all cases the ratio between the frequencies  $\Omega_1$  and  $\Omega_2$

and  $\omega_{m_0}$  of the external field was the same,  $\Omega_1 : \omega_{m_0} : \Omega_2 = 7 : 8 : 9$ .

One should stress that the quantum Arnol'd diffusion takes place only in the case when the number  $M_s$  of energy stationary states in the separatrix layer is relatively large. For the first time, this point was noted by Shuryak [17] who studied the quantum-classical correspondence for nonlinear resonances. In this connection we have estimated the number of the energy states that occupy the separatrix layer. We have found that for  $a = 0.01$  the number  $M_s$  of stationary states inside the separatrix chaotic layer is more than 10, therefore, one can speak about a kind of quantum chaos in this region. On the other hand, with a decrease of the goffer amplitude, the number  $M_s$  decreases and for  $1/\sqrt{a} \approx 20$  it is of the order of one. For this reason the last right point in Fig. 9 corresponds to the situation when the chaotic motion along the coupling resonance is completely suppressed by coherent quantum effects. This effect is known as the “Shuryak border” [17]) that has to be taken into account when considering the conditions for the onset of quantum chaos.

Since the diffusive motion along the coupling resonance is effectively one-dimensional, one can naturally expect an Anderson-like localization. Indeed, the variance of the QE eigenstates of the evolution operator is finite in the  $q$ -space. This means that eigenstates are localized, and the wave packet dynamics along the separatrix layer has to reveal the saturation of the diffusion. More specifically, one expects that the linear increase of the variance of the energy ceases after some characteristic time.

In order to observe the dynamical localization in our model (along the coupling resonance and inside the separatrix layer), one needs to study long-time dynamics of wave packets. Our numerical analysis for large times (see curves (c) on Fig. 8) has confirmed that after some time  $t \sim 200T$ , the diffusion-like evolution stops for different values of the amplitude  $a$ . For larger times, the variance  $\Delta_q$  starts to oscillate around its mean value.

This effect, known as the *dynamical localization*, has been discovered in [18, 19] for the kicked rotor, and was studied later in different physical models (see, for example, Ref. [4] and references therein). One should note that the dynamical localization is, in principle, different from the Anderson localization, since the latter occurs for models with random potentials. In contrast, the dynamical localization happens in dynamical (without any randomness) systems, and is due to the interplay between (weak) classical diffusion and (strong) quantum effects.

## 5 Summary

Let us now summarize our main results. First, we have studied stationary quantum states for a particle moving inside a 2D rippled billiard. Main attention was paid to the states corresponding to the coupling resonance  $\eta = 1$ . We have numerically analyzed the structure of these states, by plotting their density distributions  $|c_{r,p}^k|^2$  in the unperturbed basis determined by the model with flat boundaries. We have found also that the patterns in  $(x, y)$ -space of the states corresponding to the inside-resonance region can be associated with the classical orbits. Also, it was shown that the states corresponding to the separatrix region, have a global structure similar to the classical orbits as well. It was found that the width of such states in the unperturbed basis is much larger than of those belonging to the inside-resonance region.

In the presence of the external two-frequency time-dependent electric field, the main interest is related to the dynamics of wave packets along the narrow stochastic regions of the corresponding classical system. Extensive numerical simulation demonstrate a kind of weak diffusion in the quantum model, that can be associated with the classical Arnol'd diffusion along the coupling resonance. We have found that the dependence of the diffusion coefficient on model parameters roughly follows the classical dependence. However, the quantum diffusion is systematically slower than the classical one. This fact manifests the influence of quantum effects.

It should be stressed that the quantum Arnol'd diffusion occurs in a deep semiclassical region, specifically for the case when the number  $M_s$  of chaotic eigenstates inside the classical stochastic layer is sufficiently large (of the order of 10 or larger). With a decrease of the amplitude of ripple, the diffusion coefficient strongly decreases, and for  $M_s \leq 1$  the diffusion disappears. Therefore, one can see how quantum effects destroy the diffusive dynamics of the wave packets.

Another manifestation of quantum effects is the dynamical localization that persists even for large  $M_s$ . Specifically, we have observed that the quantum diffusion occurs only for finite times. On a larger time scale the diffusion ceases and after some characteristic time it terminates. This effect is similar to that discovered in the kicked rotor model [18], and found later in other physical systems (see, for example, Ref. [4] and references therein). In our case the dynamical localization arises for a weak chaos that occurs inside the separatrix layer, in contrast to previous models with a strong (global) chaos

in the classical description.

The effects discussed in this communication can be observed experimentally, e.g. in semi-metal structures. Apart from the experiments with a 2D electron gas in GaAs/AlGaAs heterojunctions [9, 10], one can consider the semi-metal rippled channel with a large number of discrete quantum levels in a deep semiclassical regime. For the observation of the dynamical localization in a response to electromagnetic radiation the channel, one can take  $d = 1 \mu\text{m}$  for the width, and  $l = 2 \mu\text{m}$  for the goffer period. Then, the dimensionless value  $a = 0.01$  corresponds to the goffer amplitude of the order 3.2 nm, i.e. to few monoatomic levels. Correspondingly, for the effective electron mass  $m \simeq 0.1m_e$  the level number  $n_0 = 400$  has the energy  $E_{n_0} \simeq 0.61 \text{ eV}$ , and the resonance frequency for the transitions between nearest states is  $\omega_{n_0}/2\pi \simeq 740 \text{ GHz}$ . In these units the period of the external field is equal to  $1.1 \times 10^{-11} \text{ s}$ , and the dimensionless value of the perturbation  $f_0 = 10$  corresponds to an electric field  $E \simeq 0.24 \text{ V/cm}$ . It should be noted that for above parameters the diffusion saturation time, corresponding to approximately 200 periods, is of the order of 2.2 ns. It is clear that the electron scattering time has to be much larger. The last condition can be realized in semi-metal Bi or Sb structures, where the mean free path is of the order of 1 mm and the scattering time is about 10 ns for the temperature  $\sim 1 \text{ K}$ .

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## References

- [1] M. Wilkinson and E.J. Austin, Phys. Rev. A **46** (1992) 64.
- [2] D.M. Basko, M.A. Skvortsov and V.E. Kravtsov, Phys. Rev. Lett. **90** (2003) 096801.

- [3] B.V. Chirikov, Phys. Rep. **52** (1979) 263.
- [4] A.J. Lichtenberg and M.A. Lieberman, Regular and Chaotic Dynamics, Springer-Verlag, New York, 1992.
- [5] V.Ya. Demikhovskii, F.M. Izrailev, and A.I. Malyshev, Phys. Rev. Lett. **88** (2002) 154101; V.Ya. Demikhovskii, F.M. Izrailev, and A.I. Malyshev, Phys. Rev. E **66** (2002) 036211.
- [6] V.Ya. Demikhovskii, F.M. Izrailev, and A.I. Malyshev, Phys. Lett. A **352** (2006) 491.
- [7] B.V. Chirikov, Atomnaya Energia **6**, 630 (1959) [Engl. Transl. J. Nucl. Energy Part C: Plasma Phys. **1**, 253 (1960)].
- [8] V.I. Arnol'd, DAN USSR **156** (1964) 9 (in Russian).
- [9] L. DiCarlo and C.M. Marcus, J.S. Harris Jr., Phys. Rev. Lett. **91** (2003) 246804.
- [10] A.G. Huibers *et. al.*, Phys. Rev. Lett. **83** (1999) 5090.
- [11] G.A. Luna-Acosta, A.A. Krokhin, M.A. Rodriguez, P.H. Hernandez-Tejeda, Phys. Rev. B **54** (1996) 11410.
- [12] G.A. Luna-Acosta, K. Na, L.E. Reichl, A.A. Krokhin, Phys. Rev. E **53** (1996) 3271.
- [13] G.A. Luna-Acosta, J.A. Mendez-Bermudez, F.M. Izrailev, Phys. Rev. E **64** (2001) 036206.
- [14] F.M. Izrailev, J.A. Mendez-Bermudez, and G.A. Luna-Acosta, Phys. Rev. E **68** (2003) 066201; F.M. Izrailev, N.M. Makarov, M. Rendon, Phys. Stat. Sol. (b) **242** (2005) 1224; Phys. Rev. B **72** (2005) 041403(R).
- [15] V.Ya. Demikhovskii, S.Yu. Potapenko, A.M. Satanin, Fiz. Tekh. Poluprovodn. **17** (1983) 213 [Sov. Phys. Semicond. **17** (1983) 137].
- [16] P.F. Goldsmith, Quasioptical Systems: Gaussian Beam Quasioptical Propagation and Applications, John Wiley&Sons Inc., 1998.
- [17] E.V. Shuryak, Zh. Eksp. Teor. Fiz. **71** (1976) 2039 (in Russian).



- [18] G. Casati, B.V. Chirikov, F.M. Izrailev, and J. Ford, Lect. Notes Phys., **93** (1979) 334.
- [19] B.V. Chirikov, F.M. Izrailev, and D.L. Shepelyansky, Sov. Sci. Rev. 2 (1981) 209.
- [20] T.M. Fromhold *et al.*, Nature **428** (2004) 726.

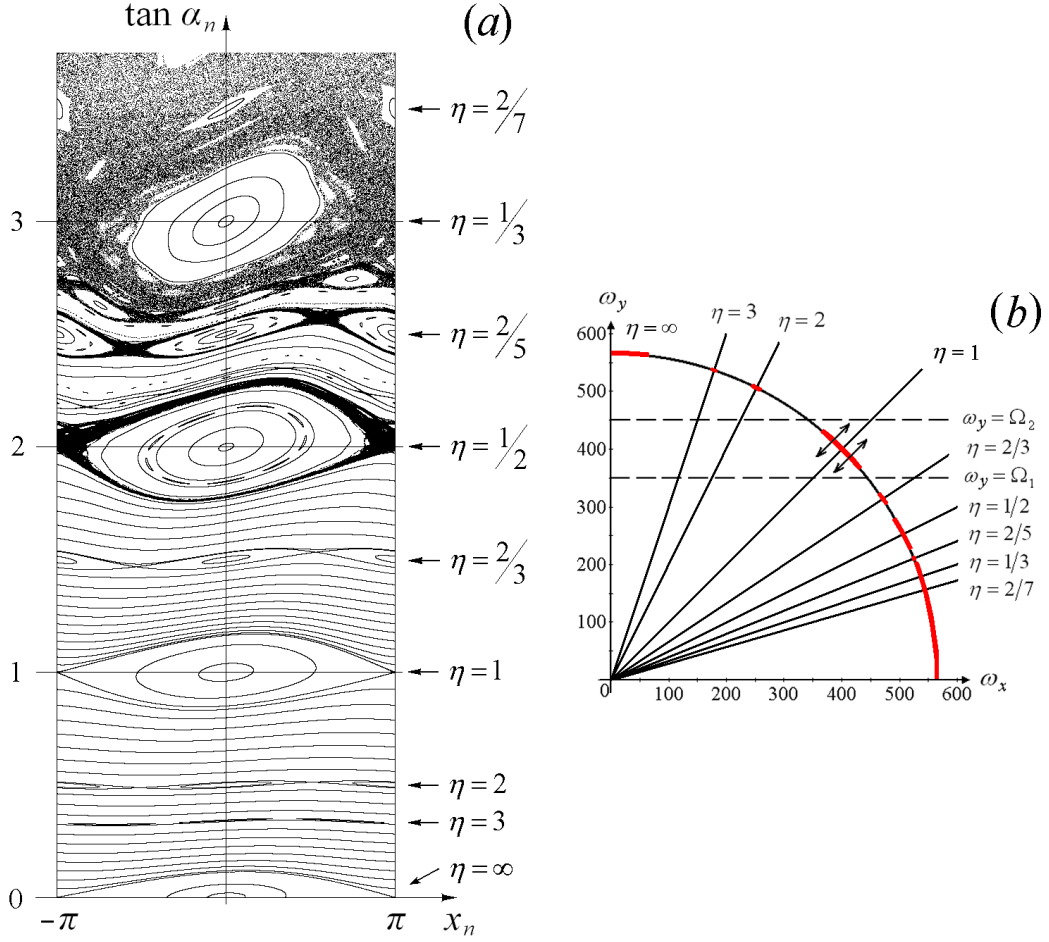


Figure 2: The resonance structure for the rippled channel model. (a) The Poincaré section is shown for  $d = \pi$  and  $a = 0.01$ , demonstrating the structure of some of coupling resonances. (b) The positions of some coupling resonances and two driving resonances (dashed lines) and the isoenergetic curve  $E = 1.6 \cdot 10^5$  on the frequency plane are shown, together with the resonances widths (by red color).

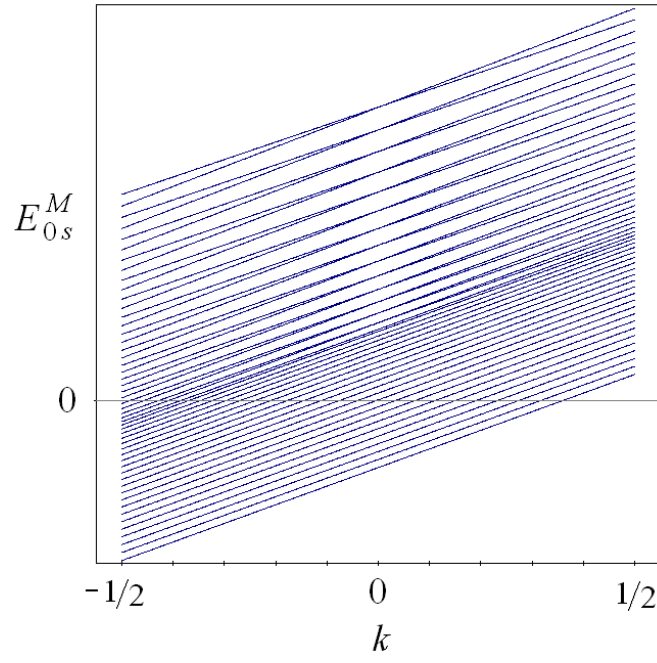


Figure 3: Structure of the energy spectrum in dependence on the Bloch number  $k$ , for  $q = 0$  and  $n_0 = m_0 = 400$ ,  $d = \pi$  and  $a = 0.003$ , see text.

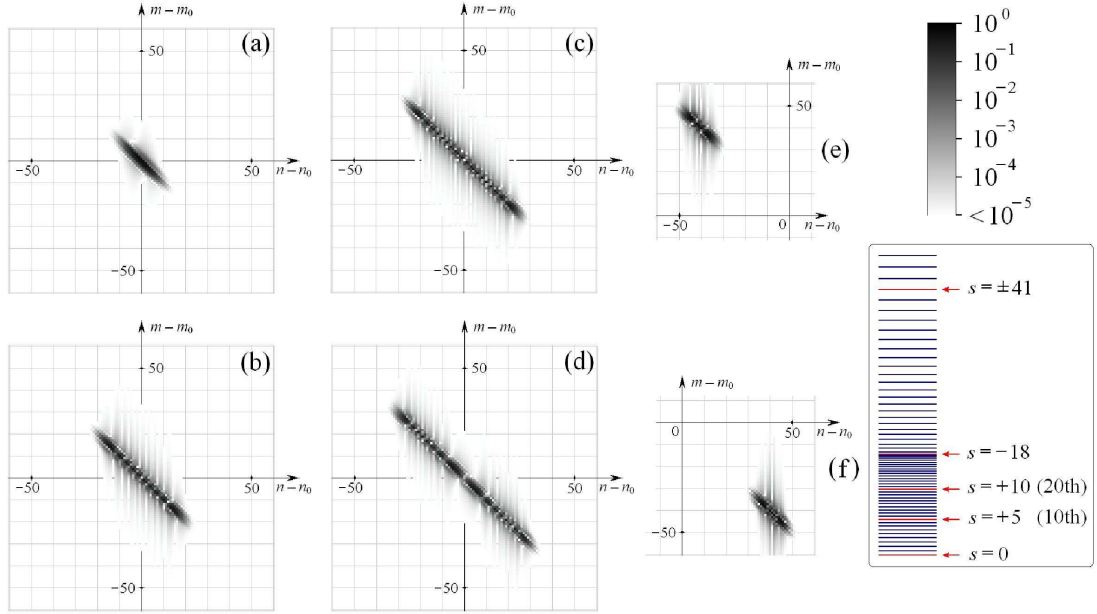


Figure 4: Probability  $|c_{rp}^k|^2$  in grayscale for different resonant quantum states ( $\eta = 1$ ) corresponding to the Mathieu-like group  $q = 0$  (see inset), for parameters  $d = \pi$ ,  $a = 0.005$  and  $k = 0.01$ , and for  $n_0 = m_0 = 400$ . (a) lowest

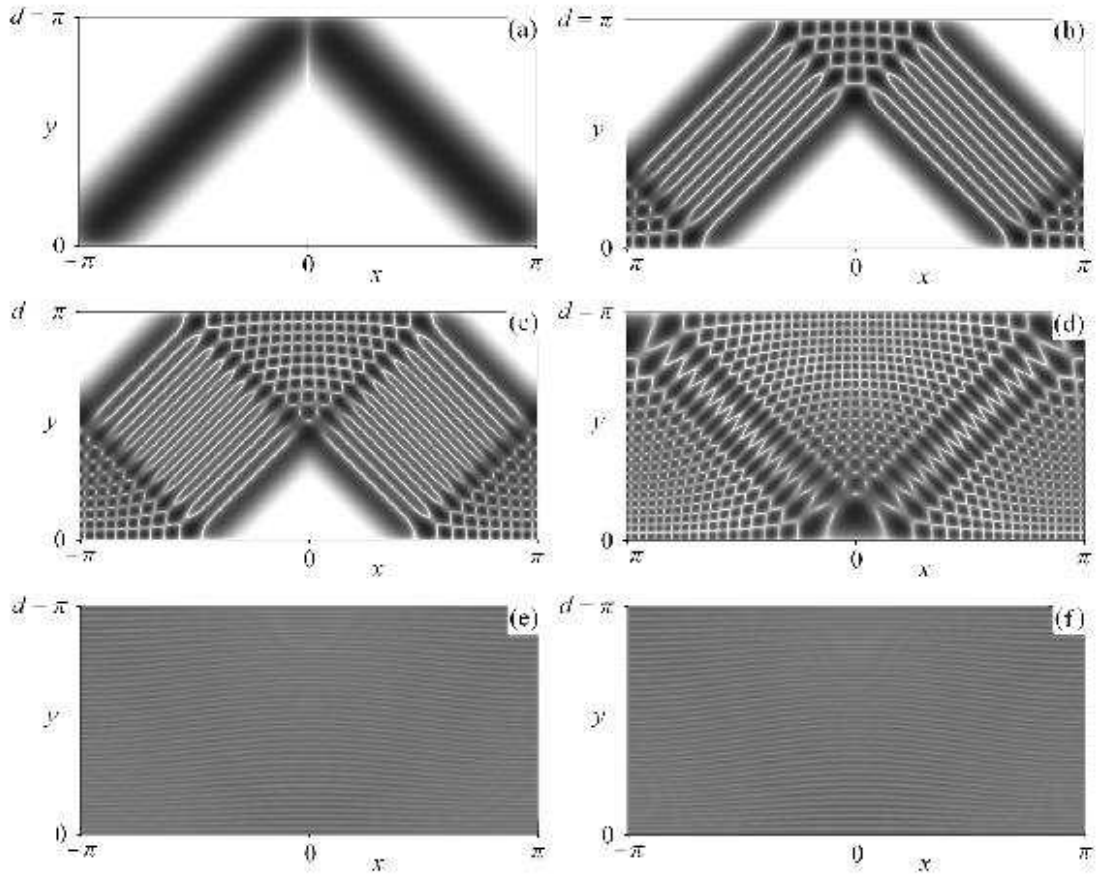


Figure 5: Examples of eigenstates probabilities  $|\psi_{q,s}^k(x, y)|^2$  in the grayscale for the same model parameters and quantum state numbers as in Fig. 4.

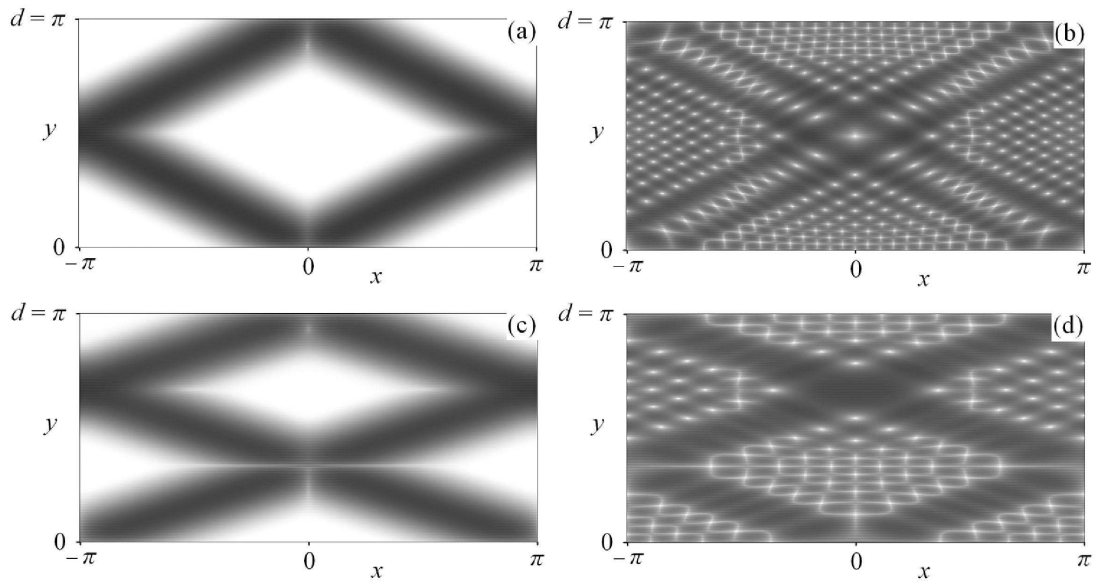


Figure 6: Examples of eigenstates probabilities  $|\psi(x, y)|^2$  (for resonance centre and for near-separatrix state) in grayscale for the same model parameters, but for the resonances  $\eta = 1/2$  (a), (b) and  $\eta = 1/3$  (c), (d).

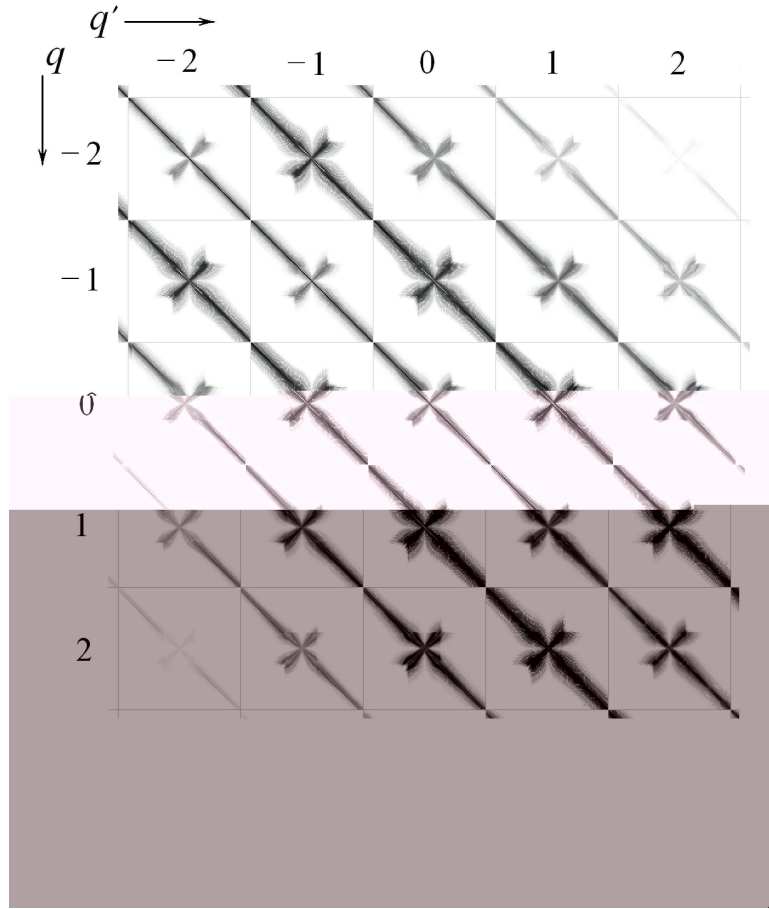


Figure 7: The distribution of the matrix elements modulus  $|U_{q,s,q',s'}(T)|$  for the same parameters as in Fig. 4 and  $f_0 = 5$ . There are  $101 \times 101$  elements in each block.

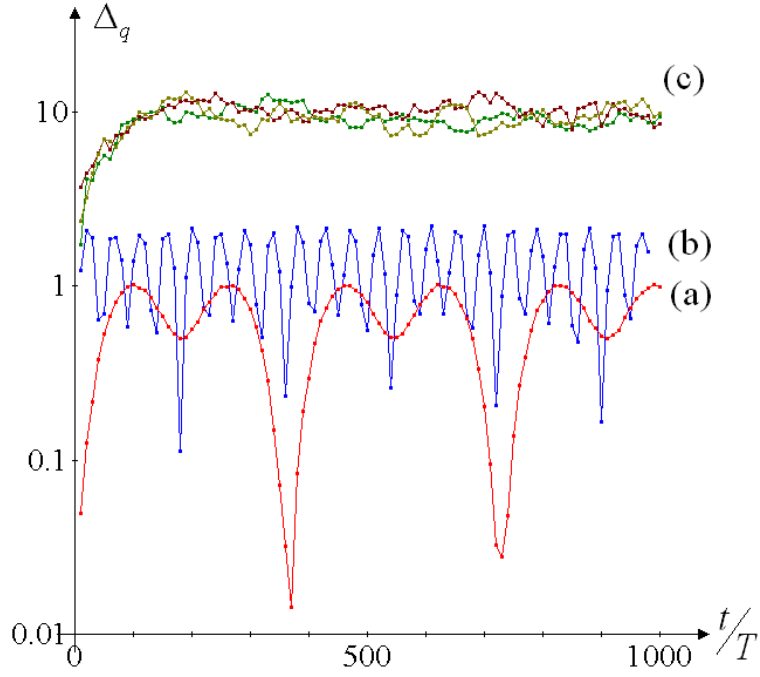


Figure 8: Time dependence of the variance  $\Delta_q$  for different initial states at the coupling resonance for the group with  $q = 0$ : (a) lowest level with  $s = 0$ , (b) above-separatrix level with  $s = +45$ , (c) near-separatrix levels with  $s = -21, +22, -23$ . Here  $a = 0.01$  and  $f_0 = 10$ .



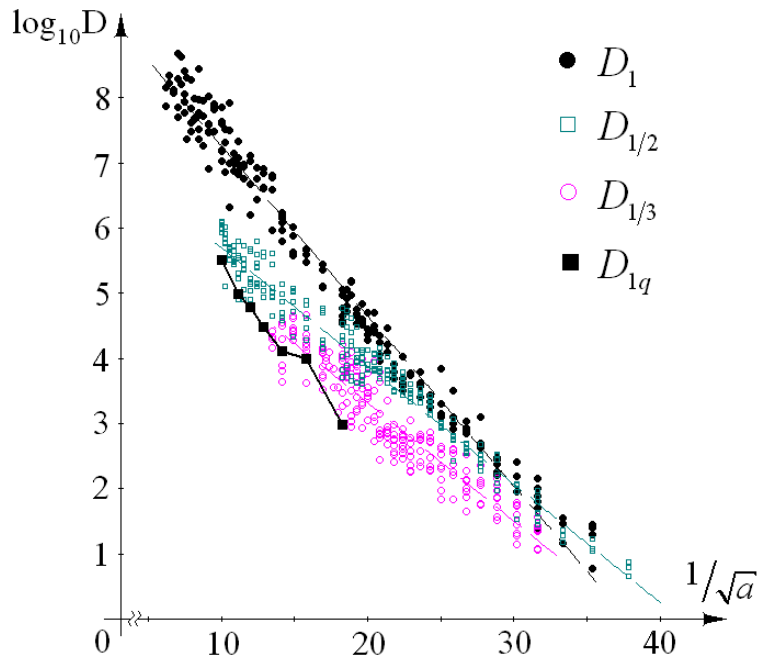


Figure 9: Classical,  $D_1$ ,  $D_{1/2}$  and  $D_{1/3}$ , versus quantum,  $D_{1q}$ , diffusion coefficients in the dependence on the amplitude  $a$  of the rippled profile.